ADVENTURES IN CELESTIAL MECHANICS

Second Edition

VICTOR G. SZEBEHELY
HANS MARK
Department of Aerospace Engineering
and Engineering Mechanics
The University of Texas at Austin

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Library of Congress Card No.:
Applied for

British Library Cataloging-in-Publication Data:
A catalogue record for this book is available from the British Library

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

Printed in the Federal Republic of Germany
Printed on acid-free paper

Printing Strauss GmbH, Mörlenbach
Bookbinding Litges & Dopf Buchbinderei GmbH, Heppenheim
Cover Photo NASA

ISBN-13: 978- 0-471-13317-9
ISBN-10: 0-471-13317-5

A TRIBUTE TO VICTOR G. SZEBEHELY

Victor G. Szebehely
1921–1997

It is with a heavy heart that I write these words. On September 13, 1997, my friend, colleague, and mentor, Professor Victor G. Szebehely, died at his home in Austin. I have to confess that it has been very hard for me to carry on with this book without him. He was the guiding spirit of the work.

Victor and I had much in common. We were both refugees from Europe—he from Hungary and I from Austria—fleeing the twin scourges of Nazism and Communism. We both became Americans and we both worked on technical projects related to the national security. We both came to love The University of Texas. Finally, we both developed a strong interest in space exploration, and Victor made important contributions to the success of our journeys to the Moon.

Where Victor was unique was in his deep understanding of celestial mechanics and his ability to apply this knowledge to the solution of practical problems. Victor did not hesitate to tackle the toughest scientific problem in his field which is the subject of the final chapter in this work: The problem of three bodies. He had the intellectual courage to take on the hardest challenges and the intellectual horsepower to make critical contributions of lasting value.

I would be remiss if I did not mention Victor's personal qualities. In addition to being a man of intellect, Victor was also a man of good will who was honored and respected by all who knew him. Perhaps most important for his friends was his impish sense of humor. We both had our offices on the fourth floor of Woolrich Hall, the aerospace building on our campus. I was on one end of the floor and he was on the other. One morning I was complaining to him about something that had gone wrong with our research funding in the Congress. Suddenly, he proposed that we resurrect the Austro-Hungarian Empire on the fourth floor and raise the Imperial banner "with appropriate salutes" every morning. "Maybe," he said, "that will solve your problem!" I laughed and promptly forgot what was upsetting me.

Victor Szebehely was a great man whose influence was widespread. I was one of the people who came into his orbit and I am proud to have been his student. I mourn him and I miss him. Rest in peace, my friend, and go with God.

September 1997 HANS MARK

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PREFACE

This volume is the second edition of *Adventures in Celestial Mechanics* published by Victor G. Szebehely in 1989 at the University of Texas Press. The subject of this edition is the same as the previous one which was to quote the earlier introduction "to study the motion of natural and artificial bodies in space." The work is still intended as a textbook for a first course in orbital mechanics and spacecraft dynamics and we have attempted to produce a second edition that maintains the spirit of the first. This was also stated succinctly in the introduction of the first edition: "fundamental ideas will be emphasized and will not be cluttered up with details that are available in the immense literature of this field."

Having described the similarities between this book and the previous one, we should say a word about the changes. The principal difference between the two editions is that we have added some material that strengthens the treatment of the "artificial bodies in space." A chapter on rocket propulsion has been added that describes what must be done to get things into space. We have included a chapter on elementary spacecraft dynamics so that we discuss not only trajectory maneuvers but also how spacecraft are stabilized and oriented. Finally, we have included a chapter on the exploration of the solar system in which the "natural" and "artificial" bodies are treated together. This area is one of the genuine triumphs of modern science and engineering, and it constitutes the most important modern application of celestial mechanics. Therefore we felt that it was necessary to address it even in an elementary course.

In addition to these major changes, there are minor ones as well. In several instances (e.g., Lambert's theorem and gravity-assist trajectories) we have included details that were not present in the first volume. We have also somewhat expanded the discussion of the three-body problem to include chaotic motion in nonlinear systems.

For the most part, therefore, the second edition is similar to the first. Each of the chapters contains some numerical examples so that students will become familiar with how various calculations are performed. Problems are also included at the end of each chapter. Finally, appropriate references are mentioned at the end of each chapter and also in the appendix.

Many people helped us to write this book. We are grateful to these colleagues in particular: Professor Roger Broucke for his help in developing the derivation of Lambert's theorem, Professors Wallace Fowler and Bob E. Schutz for their help in writing and revising Chapter 9 (Elements of Spacecraft Dynamics), and Professor Raynor L. Duncombe for carefully reading and commenting on the manuscript. We owe a very special debt of gratitude to Ms. Maureen A. Salkin who did a superb job typing the entire manuscript. In addition, Ms. Salkin made important editorial suggestions that significantly improved the quality of the work. Finally, we would like to thank all of the students who were in our classes during the years that we have taught this course at The University of Texas at Austin. These young people provided us with continuing stimulation and inspiration which made it a great pleasure for both of us to work on this project.

VICTOR G. SZEBEHELY
HANS MARK

Austin, Texas
August 1997

CHAPTER 1

ON THE SHOULDERS OF GIANTS: AN HISTORICAL REVIEW

People have looked at the stars since the dawn of history. The obvious "permanence" of the heavens and the regularity of the motions executed by the Sun, the Moon, and the planets soon led people to look for explanations. Each of the major civilizations produced a "cosmology" that was based on more or less crude observations and was melded with the myths of the civilization. These "theories" of the cosmos were important in that they were early attempts to understand how the universe works. While many of these had philosophical and perhaps literary value, they lacked what is essential in a modern scientific theory: predictive value. None of these theories were able to make really accurate predictions of phenomena such as eclipses or were able to explain why the observed regularities in the planetary motions exist.

During the fourth and third centuries before the birth of Christ, there was a great flowering of civilization in Greece. Philosophical schools were established by a number of people, and one of the major topics of interest was cosmology. Many theories were set forth, including at least one that put the Sun at the center of the solar system. Aristarchus of Samos (ca. 270 B.C.) developed some clever techniques for measuring both the sizes and the distances to the Moon and to the Sun. Although this methods were crude, and in the case of the Sun somewhat flawed, he did conclude from his observations that the Sun must be much larger than Earth. It was from this "measurement" that Aristarchus was the first to conclude that the Sun, rather than Earth, should be placed at the center of the solar system. At about the same time, Eratosthenes of Alexandria (ca. 276 B.C.) actually measured the radius of Earth by comparing the length of the shadow cast by similar vertically placed rods, one in Syene and the other at Alexandria, at high noon on the first day of summer. The value he calculated was within 20% of the ones obtained by modern measurements.

However, by far the most influential natural philosopher of the period was Aristotle (384–322 B.C.). He taught that the only way to understand the world was by the application of pure *reason*. This approach led him to two conclusions that were to impede progress for more than 18 centuries. Aristotle argued that it was *common sense* to conclude that Earth is fixed in space and located at the center of the universe. Furthermore, he said that the gods lived in heaven, and thus all motion in the heavens had to be "perfect," by which he meant uniform and circular. Unlike Aristarchus, most philosophers of the day did not attach much value to detailed observations and measurements. Thus, Aristotle's views prevailed because of his enormous influence; he was, after all, the teacher of Alexander the Great.

The cosmology of Aristotle was developed in a systematic way by Claudius Ptolemaeus (ca. A.D. 140). Ptolemaeus was a Greek who lived in Alexandria, where he produced a monumental treatise called the Almagest that included a detailed section on cosmology. He placed Earth at the center of the universe and said that the stars were fixed on a large sphere that rotated around the central Earth once every day. Since the Sun, the Moon, and the planets all moved relative to the stars, they were said to be attached to different spheres, all rotating in uniform motion around Earth. To explain the complex (and sometimes even retrograde) motion of some of the planets, smaller spheres were attached to the larger ones, and the planet was then located on the surface of the small sphere. This sphere also would rotate with uniform angular velocity, thus preserving the Aristotelian doctrine of uniform circular motion for this complex system of spheres upon spheres. Using what were called *cycles* and *epicycles,* this model turned out to be remarkably accurate given the state of astronomical instruments in the second century A.D. While the model of cycles and epicycles had descriptive value, it did not explain why the stars and planets moved the way they do.

It took more than a thousand years to change this stage of affairs. In the thirteenth century, Roger Bacon, an English cleric, was the first to propose that hard knowledge (theories, if you will) must be based on observation and that these observations must be rigorously controlled and objective; that is, they must be repeatable by any observer. What we now call the "modern" science slowly evolved from Bacon's ideas. In a very real sense, Bacon was the one who set the stage for the great scientific achievements of the renaissance period.

It can be argued that the very first important and genuine application of the modern scientific method was the complete and detailed understanding of how the solar system works. All of the hallmarks of how modern science is done are there: the introduction of a new hypothesis, perhaps even for the wrong reason; the development of a reliable body of measurements; the rejection of the existing theory by showing that the measurements support the new hypothesis; and, finally, the demonstration that the new theory can explain things that could not be understood previously. The first tentative steps were taken by Nicolaus Copernicus (1473–1543) (Mikolaj Kopernik in Polish), who introduced the hypothesis of a solar system with the Sun, rather than Earth, at the center. In the first instance, he did this for a practical reason, since it was an attempt to simplify the calculations necessary to maintain an accurate calendar. Using the older, geocentric model of the solar system developed by Claudius Ptolemaeus (Ptolemy), calendar calculations had become very complicated as better measurements became available. Copernicus nursed the hope that, by placing the Sun at the center of the solar system, he could reduce the number of parameters necessary to make good predictions of the celestial phenomena and events that determined the calendar. In this effort, Copernicus was only partially successful. However, what is important is that a "truth" dawned on him during the process of his work, which was that the Sun really is located at the center of the solar system. As a conservative clerical lawyer, Copernicus was shocked by his own hypothesis, and he never published anything that contained the absolute assertion that the Sun was at the center of the solar system during his lifetime. His major work, "De Revolutionibus Orbium Coelestium" was published only after his death. We thus have the accidental stumbling on a major "truth" that occurs so often in the modern scientific process.

A second feature of scientific discovery is accurate and reliable experimentation. Tycho Brahe (1546–1601) was the most important exponent of this process of understanding the solar system. Tycho was a Danish aristocrat who received a cosmopolitan and international education. He took up observational astronomy as a hobby and, because of his great wealth, was able to build what was, for his time, the finest astronomical observatory in the world. It was called the Uranienborg (castle of the sky) and was located on the Island of Hven near Copenhagen. Tycho, for the first time, made accurate measurements of the positions of the Sun, the Moon, the planets, and the stars. What is more important is that he made these observations systematically over more than 20 years. He was therefore the first to produce accurate ephemeris tables, and as we shall see, these eventually turned out to be of decisive importance. Three years before his death, Tycho was forced to leave Denmark. The Emperor Rudolf II then invited Tycho to become the Astronomer to the Imperial Court in Prague. It was there that he met Johannes Kepler, which led finally to the great breakthrough.

Galileo Galilei (1564–1642) also made a most important "experimental," or observational, discovery by being the first person to turn the newly invented telescope toward the sky. By observing that the four large moons of Jupiter execute more or less circular orbits around the planet, he had discovered a small system that demonstrated clearly how the larger solar system works. It was this observational discovery that provided a convincing argument that the Copernican hypothesis regarding the position of the Sun at the center of the solar system was correct. The contributions of both Tycho and Galileo were critical: Galileo's was qualitative, but it gave others the courage to go ahead. Galileo was also an enthusiastic and articulate controversialist and he was able to engage the educated public in the cosmological debate. It is interesting that the great work of Copernicus, "De Revolutionibus Orbium Coelestium" was put on the Index by the Vatican in 1616 (70 years after publication), only after Galileo began his propaganda campaign for the Copernican system. Finally, it was Tycho Brahe who provided the trustworthy numbers.

As important as these contributions were, the real intellectual breakthrough came from Johannes Kepler (1571–1630). Kepler was the son of a German noncommissioned officer. His talents in mathematics were recognized very early in his life, and he was educated by the local clerical authorities. Eventually, he was appointed Professor of Mathematics at the University of Graz in Austria, where he began his astronomical studies. He believed in the heliocentric hypothesis, and he made several attempts to develop a mathematical model of the solar system based on placing the Sun at the center. All of these models failed to fit the observations, and so, in 1599, he decided that he would go to work for the man who had the best observations, Tycho Brahe. Tycho had been exiled from his native Denmark in 1598 and had moved to Prague. Kepler applied for the post as Tycho's assistant at the Imperial Court and his application was accepted. Unfortunately, Tycho died shortly after Kepler arrived in Prague, and Kepler was forced to fight a lengthy legal battle to get access to Tycho's ephemeris tables. Eventually, he succeeded and this is when his great work began.

Perhaps the single most difficult thing that must be done in the process of scientific discovery is to abandon that which was previously taken to be the "truth." Habits of thinking are hard to break, but this is exactly what Kepler did when he abandoned the old Greek idea enshrined by Aristotle that all heavenly bodies must execute perfect motion, meaning that their motion must be in circular orbits moving at uniform speed. In doing his calculations, Kepler could not explain Tycho Brahe's observations of the motion of Mars with the assumption that Mars was moving in a uniform circular orbit around the Sun. It was at this point that Kepler made his great breakthrough. He chose to abandon Aristotle and to believe the observations of Tycho Brahe and turned the question around: Given the observations of Tycho, what kind of orbit does Mars execute? It was in answering this question that Kepler discovered his quantitative laws of planetary motion. A good argument can be made that Kepler's step was actually the most difficult one in the entire process, because he had to do two things that involved great intellectual risks. First, he had to abandon the centuries-old idea of uniform circular motion and, second, he had to believe Tycho's observations to derive his laws. It was the complete rejection of the old and the leap of faith in the new measurements that made Kepler's achievement the most remarkable one in the entire story.

Kepler's laws of planetary motion may be stated as follows:

1. Planets move around the Sun in elliptic orbits with the Sun located at one focus of the ellipse.
2. As the planet moves in its orbit around the Sun, equal areas as measured from the focus are swept out in equal times. (This law implies that the planet moves more rapidly when it is close to the Sun compared to when it is farther away.)
3. The square of the period of the orbit is proportional to the cube of the semimajor axis of the elliptic orbit.

The final chapter in this history came when Isaac Newton realized that Kepler's laws were the consequences of more basic principles, the law of universal gravitation and the so-called second law of motion, which relates the acceleration of an object with the force that is applied to move it. These two principles were sufficient to explain Kepler's laws and much else as well. If Kepler was the one who broke with the past, it was Newton who looked to the future. As Newton put it, "If I have been able to see a little farther, it is because I stood on the shoulders of giants."

Isaac Newton was born at Woolsthorpe in Lincolnshire on Christmas Day in 1642. He died almost 85 years later in 1727. He received his B.A.

degree in Cambridge in 1665. In 1669, when his professor Isaac Barrow resigned, he requested that Newton be given his professorship. Newton's complete dedication to his work resulted in headaches, sleepless nights, irregular eating habits, and finally a nervous fatigue at 50 years of age. He mentions these problems in his notes on the computations of the motion of the Moon.

In 1687, before switching to administrative activities as the Warden and, in 1699, as the Master of the Mint, his book, entitled *Philosophiae Naturalis Principia Mathematica,* was published by the Royal Society of London. It is interesting to see how dynamical problems can become complicated at Newton's insistence that they be solved using geometry instead of calculus. This makes the *Principia* a hard book to read and leads to the question of why Newton, one of the inventors of calculus, did not use calculus in his book. Newton had used calculus to formulate and solve some of the problems presented in the *Principia* but, being afraid of criticism, described his work using geometry.

Newton's conflicts with Leibnitz concerning the discovery of calculus are well represented in the literature, and this may be another reason why geometry dominates the *Principia.* Their controversy regarding the deterministic nature of dynamics and celestial mechanics is less known. Today, Newtonian mechanics is sometimes erroneously associated with complete predictability in dynamics, which was Leibnitz's dogma and was not accepted by Newton. At this point, Laplace's demon enters the picture: knowing all initial conditions and all laws of nature and predicting the future. Laplace takes the side of Leibnitz. (See the list of bibliography at the end of this chapter.)

In 1665, because of plague, Newton left Cambridge and went back to his birthplace, where he could work undisturbed. The unverified apple incident, which could have happened here, describes the importance of connecting seemingly unrelated phenomena; in this case, falling stones (or apples) on the one hand and planetary motion on the other. In fact, Newton describes the idea leading to artificial satellites with the following thought experiment: If stones are thrown from the top of a mountain with small horizontal velocities, they will hit the ground, but as the velocity is increased, circular and elliptic orbits are obtained around Earth. It was here, amid conditions of creativity, concentration, and peace, that Newton arrived at the general theory of gravitation.

Newton became the president of the Royal Society at the age of 60 and was knighted by Queen Anne in 1705. He died in 1727 and is buried at Westminster Abbey in London.

Since, in this book, we wish to concentrate on dynamics and celestial mechanics, for a description of Newton's many other significant scientific contributions (e.g., his *Opticks*, published in 1704), the reader is referred to the literature.

Since Newton's laws of dynamics and his law of gravitation will be described here, a few general historical comments might be appropriate.

Newton's three laws of motion, forming the basis of dynamics, are as follows:

1. Every body perseveres in its state of rest or uniform straight-line motion unless it is compelled by some impressed force to change that state.
2. The change of motion is proportional to the motive force impressed and takes place in the same direction as the force.
3. Action is always contrary and equal to reaction.

There are many variations of these laws, some by Newton himself, who made changes and corrections. Also, differences exist in the literature as the laws were translated from the original Latin text. Once again, the soundest language, mathematics, comes to our aid. Using the concept of linear momentum (which Newton called motion), we can express the first and second laws by the equation

$$\frac{d(m\mathbf{v})}{dt} = \mathbf{F} \tag{1.1}$$

Note that Newton did not mention acceleration when giving his laws of motion. For a constant value of the mass, the above equation should read: $m(d\mathbf{v}/dt) = \mathbf{F}$. Our textbooks use the concept of acceleration and give Newton's law as $m\mathbf{a} = \mathbf{F}$. This is of less generality than Newton's original formulation, which is applicable to variable mass and, therefore, for rocket propulsion.

Newton's law of gravitation, as discussed in his *Principia*, was mentioned before. The gravitational force acting between two bodies of mass m and M is proportional to the product of the masses and inversely proportional to the square of the distance between them. In vector form

$$\mathbf{F} = \frac{GmM}{|\mathbf{r}|^3}\,\mathbf{r} = G\frac{mM}{r^2}\,\hat{r} \tag{1.2}$$

where $\hat{r}$ is the unit vector pointing in the direction $\mathbf{r}$ and G is the gravitational constant that determines the "strength" of the gravitational field.

Probably nothing describes Newton better than one of his own statements: "I seem to have been only like a boy playing on the seashore and diverting myself now and then finding a smoother pebble or a prettier shell than ordinary, while the great ocean of truth lay all undiscovered before me."

It is a common error to believe that the behavior of the solar system and the rules of orbital mechanics were completely understood as a result of the work of Isaac Newton. He took a giant step, but many critical questions remained unanswered. Newton solved what we call the "problem of two bodies," which means that he developed the means to predict the motion of two bodies interacting through the gravitational field. For a system of more than two bodies, Newton's equations cannot be solved. Fortunately, the solar system is dominated by the Sun, which accounts for more than 99.8% of its entire mass. Thus, to a very good approximation, the motion of each planet can be calculated *as if* only the Sun and that planet counted. Thus, Newton was able to deduce his laws. With the advent of more accurate astronomical measurements in the eighteenth century, discrepancies appeared that could only be explained by taking into account the effects of the other planets in the solar system.

Following Newton's work, several brilliant astronomers and mathematicians used Newton's laws and methods to attack a number of important problems. The first of these was Edmund Halley (1656–1742), who observed and calculated the orbit of the comet named after him using Newton's laws of motion. Studying several cometary orbits, he established the facts that, contrary to planetary orbits, some comets had large angles of inclination and that some had periodic orbits. Halley's contributions were numerous and important to celestial mechanics, but his insistence on and support of the publication of Newton's *Principia* probably represent his greatest influence on today's celestial mechanics.

The Swiss-born mathematician Leonhard Euler (1707–1783) was a student of Johann Bernoulli. In 1727 Euler went to St. Petersburg in Russia for 14 years and was associated there with the Imperial Academy. From there, at the invitation of Frederick the Great, he went to Berlin, where he remained for 25 years. He returned to St. Petersburg at the invitation of the czarina, Catherine the Great, in 1766.

Euler's work on the motion of the Moon was of considerable interest to Catherine the Great as his lunar tables and his second lunar theory, published in 1772 under the title *Theoria Motuum Lunae* in the Communications of Petropolis, helped the navigation of ships in the Russian Navy. Before it appeared in its published form, his lunar theory was used by the Astronomer Royal, Nevil Maskelyne, in the British Nautical Almanac as the basis for the lunar ephemeris. These tables were first published in 1767 and were used by the British Navy for navigation. (These were probably the first, but certainly not the last, uses of celestial mechanics by the military.)

Newton's most important successors, who truly extended his methods, were two Frenchmen whose lives spanned the last years of the eighteenth century and the first years of the nineteenth: Joseph Louis Lagrange (1736–1813) and Pierre Simon de Laplace (1749–1827). Lagrange was born in Turin, Italy, where he was appointed professor of geometry at the artillery academy at the age of 19. In 1766, he went to Berlin, filling Euler's vacated position at the invitation of Frederick the Great, where he spent 20 years. The next invitation came from Louis XVI to Paris, where he became professor at the École Polytechnique in 1797. His apartment in Paris was in the Louvre; he was buried in the Pantheon.

Lagrange's announcement concerning the triangular libration points in the Sun–Jupiter system and his prediction of the possible existence of asteroids in these regions date from 1772. Observational astronomers did not verify the existence of these bodies for another 134 years. In this case, theory was certainly ahead of observation. His work on the solar system using the method of variation of parameters (1782) is one of the fundamental contributions in celestial mechanics.

Lagrange's celebrated *Mécanique Analytique* was published in 1788.

Laplace was born in Beaumont-en-Auge and became professor at the École Militaire in Paris at the age of 18. One of his major contributions concerned the stability of the solar system (1773, 1784), for which he developed the methods of perturbation theory to solve the many-body problem. After a lengthy series of calculations, he concluded that the solar system was indeed stable and that Newton's famous "clockwork universe" really existed. As things turned out, Laplace was wrong, and the problem of "stability" is still unsolved. Laplace also introduced the concept of the potential function and what is known today as Laplace's equation (1785). His lunar theory, published in 1802, followed Euler's. The five volumes of his *Mécanique Céleste* were published between 1799 and 1825.

Although the perturbation methods introduced by Laplace did not yield an answer to the stability question, they were extremely useful in making more accurate calculations of the behavior of planets, comets, and asteroids. The most spectacular application of perturbation theory was the discovery of the eighth planet, Neptune, because of the small perturbations the planet causes in the motion of the planet Uranus. John Couch Adams and U. J. J. Leverrier performed these calculations in 1845 and predicted the position of Neptune. In the next year, J. F. Encke and H. L. d'Arrest found Neptune essentially where it was supposed to be. In the early years of this century, Percival Lowell and William H. Pickering tried to do the same thing by looking at small perturbations in the orbit of Neptune. The theoretical work done by Lowell and Pickering between 1910 and 1917 was detailed and extensive. Lowell died in 1917, but Pickering continued to work on the problem. Eventually, another search for a trans-Neptune planet was initiated, and in 1930, the young astronomer Clyde W. Tombaugh discovered Pluto. The "predictions" of Lowell and Pickering could not have had anything to do with the discovery of Pluto since the planet turned out to be much too small to affect Neptune in the way Lowell and Pickering had calculated. In any event, these remarkable achievements effectively completed the inventory of planets in our solar system. They were stimulated by the development of perturbation theory.

The most important contributor to celestial mechanics in the final years of the nineteenth century and the early years of the twentieth was another Frenchman, Henri Poincaré (1854–1912). He was one of the most prolific writers in the field of mathematics and celestial mechanics, contributing more than 30 books and 500 memoirs. The three volumes of his *Méthodes Nouvelles de la Mécanique Céleste* appeared in 1892, 1893, and 1899 and have been recently translated into English by NASA. This was followed by his *Léçons de Mécanique Céleste* in 1905–1910. Concentrating on the problem of three bodies, Poincaré established the concept of nonintegrable dynamical systems. His theorem seriously affected the results of workers who intended to show the stability of the solar system by representing the orbital elements of the planets in Fourier series. Since these series, in general, are conditionally convergent or divergent according to Poincaré's theorem, the "solutions" do not show stability. Thus Laplace's conclusion of a century earlier was shown to be wrong. Poincaré's work also provided the first instance of what is now called "deterministic chaos." The problem of three bodies is described by a complete set of deterministic equations. Yet, the behavior of the three-body system may become "chaotic," which in this case means unpredictable, under certain conditions. It may very well be that this will turn out to be Poincaré's lasting legacy.

In recent years, a most significant development has furthered the science and engineering of orbital operations and that is the advent of artificial satellites and spacecraft. The demands of space navigation have clearly been a major factor in the recent progress of celestial mechanics. This effort has been greatly enhanced by the advent of high-performance digital computers, which make the approximation methods mentioned earlier less necessary. The truly fabulous accuracy of spacecraft navigation would not be possible without high-speed digital computers. For example, to put the *Pioneer 11* spacecraft into the correct trajectory around Jupiter so that it would fly past Saturn some years later required a navigational accuracy of better than one part in 10 million.

Finally, there are some very important scientific questions that are still open. Is the solar system ultimately stable? This question has not been answered in a rigorous mathematical sense. Once again, numerical methods are critical to research this question. Related to the question of stability is that of chaotic motion. Can the "Earth crossing" asteroids be explained using the principles of chaos theory? Thus, orbital and celestial mechanics, even though it is the oldest field in "modern science," still presents problems that are at the very frontier of knowledge.

What is clear is that celestial mechanics is a living field and more research is certain to reveal important and even startling new results.

The reader interested in historical details will enjoy some of the books listed in the Appendix: Andrade (1954); Bate, Mueller, and White (1971); Beer and Strand (1975); Koestler (1959); and Lerner (1973). In addition, *Men of Mathematics,* by E. T. Bell, Simon & Schuster, New York (1937); *The Great Ideas Today,* edited by R. M. Hutchins and M. J. Adler, Encyclopaedia Britannica, Inc. (1973); *From Galileo to Newton,* by A. R. Hall, Dover, New York (1981); and *The Space Station,* by H. Mark, Duke University Press, Durham, North Carolina (1987), are recommended. Regarding nondeterministic dynamics and uncertainties in celestial mechanics, see J. Lighthill's "The Recently Recognized Failure of Predictability in Newtonian Dynamics," *Proceedings of the Royal Society,* Vol. A407, pp. 35–50, 1986, and I. Prigogine's (1980) book listed in the Appendix.

For additional fascinating details of the early history, see "Copernicus and Tycho," by O. Gingerich, *Scientific American,* Vol. 229, No. 6, pp. 86–101, 1973. For Newton's contributions to cosmology, see *The First Three Minutes,* by S. Weinberg, Bantam Books, New York (1977).

Chronological List of Major Contributors to Celestial Mechanics

Aristotle,	384–322 B.C.	I. Newton	1642–1727
C. Ptolemaeus	100–178	G. W. Leibnitz	1646–1716
N. Copernicus	1473–1543	E. Halley	1656–1742
T. Brahe	1546–1601	L. Euler	1707–1783
G. Galilei	1564–1642	A. C. Clairaut	1713–1765
J. Kepler	1571–1630	J. D'Alembert	1717–1783
R. Descartes	1596–1650	J. H. Lambert	1728–1777

J. L. Lagrange	1736–1813
W. F. Herschel	1738–1822
J. E. Bode	1747–1826
P. S. Laplace	1749–1827
M. Legendre	1752–1833
K. F. Gauss	1777–1855
S. D. Poisson	1781–1840
J. F. Encke	1791–1865
G. G. DeCoriolis	1792–1843
J. F. W. Herschel	1792–1871
P. A. Hansen	1795–1874
K. G. J. Jacobi	1804–1851
W. R. Hamilton	1805–1865
U. J. J. Leverrier	1811–1877
C. E. Delauney	1816–1872
J. C. Adams	1819–1892
D. Airy	1835–1981
S. Newcomb	1835–1909
T. N. Thiele	1838–1910
G. W. Hill	1838–1914
F. F. Tisserand	1845–1896
H. Bruns	1848–1919
G. H. Darwin	1845–1912
J. H. Poincaré	1854–1912
C. V. L. Charlier	1862–1934
P. Painleve	1863–1933
E. W. Brown	1866–1938
C. Burrau	1867–1944
E. Stromgren	1870–1947
P. H. Cowell	1870–1949
W. De Sitter	1872–1934
F. R. Moulton	1872–1952
T. Levi-Civita	1873–1941
K. F. Sundman	1873–1949
E. T. Whittaker	1873–1956
H. C. Plummer	1875–1946
W. Hohmann	1880–1945
G. A. Shook	1882–1954
G. D. Birkhoff	1884–1944
W. M. Smart	1889–1975
G. E. Lemaitre	1894–1966
C. L. Siegel	1896–1981
Y. Hagihara	1897–1979
N. D. Moiseev	1902–1955
D. Brouwer	1902–1966
W. J. Eckert	1902–1971
A. Wintner	1903–1958
A. N. Kolmogorov	1903–1987
G. N. Duboshin	1904–1986
G. P. Kuiper	1905–1973
G. M. Clemence	1908–1974
P. Herget	1908–1981
E. L. Stiefel	1909–1987
S. Herrick	1911–1974
G. A. Chebotarev	1913–1975
H. Pollard	1919–1985
G. Colombo	1920–1984

CHAPTER 2

CIRCULAR ORBITS

In the preceding chapter, in equations (1.1) and (1.2), we defined the laws of motion first developed by Isaac Newton and his universal law of gravitation. It is the combining of these two laws that permits us to calculate the orbit of one body moving around another one under the influence of the gravitational interaction between the two bodies. A particularly simple case to treat is that of circular orbits. We shall assume for the moment that circular orbits are both possible and stable in the gravitational field defined by equation (1.2). This proposition will be proven in subsequent chapters.

The law of gravitation as shown in equation (1.2) is given as

$$\mathbf{F}_G = -\frac{GMm}{r^2}\hat{r} \tag{2.1}$$

where $\mathbf{F}_G$ is the force of gravity between the masses M and m. The unit vector $\hat{r}$ points in the direction of the line joining the masses, and r is the distance between the masses m and M. The situation is illustrated in Figure 2.1. For the time being, we shall assume that the masses m and M are point masses. We shall show shortly that for spherically symmetric objects the gravitational field external to the object acts as if the mass were concentrated at the geometric center of the object. The constant G is called the *gravitational constant*, and it determines the strength of the gravitational field. In Figure 2.1, we have assumed that the mass M is lo-

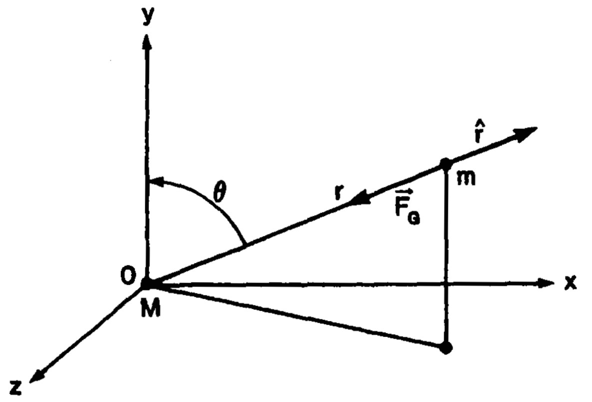

FIGURE 2.1

cated at the origin O of the coordinate system and that it is fixed in space. (We shall soon show that this is equivalent to saying that M is very much larger than m.) Note that the first $\mathbf{F}_G$ points toward the origin, where mass M is located. This happens because the gravitational force is always attractive. Note that the convention of polar coordinate systems requires that the unit vector $\hat{r}$ always points away from the origin. This accounts for the negative sign on the right side of equation (2.1), because $\mathbf{F}_G$ and $\hat{r}$ always point in opposite directions.

Figure 2.2 shows the circular orbit that we have assumed is possible in this case.

We assume that the radius of the circular orbit is R and that the vector

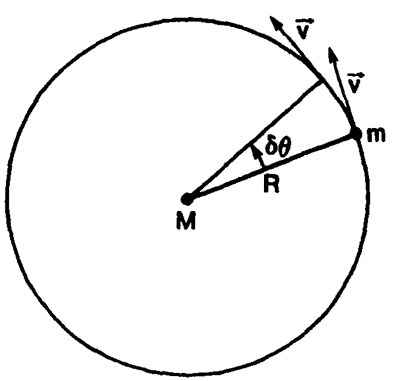

FIGURE 2.2

v is the velocity of the mass m as it moves around the mass m in the circular orbit. There are two forces acting on the mass m: the gravitational force, which points toward the mass M, and the centrifugal force experienced by an object traveling in a circular orbit. If the masses M and m were connected by a string, then the tension in the string would replace the gravitational force and would also be balanced by the centrifugal force.

The centrifugal force can now be calculated using equation (1.1) of the previous chapter:

$$\mathbf{F}_c = m\frac{d\mathbf{v}}{dt} \tag{2.2}$$

We now need to evaluate the rate of change of the velocity ($d\mathbf{v}/dt$) that appears in equation (2.2). To do that, we shall look at what happens to the orbital velocity vector. Since the gravitational force defined in equation (2.1) on the mass m is constant and since the radius of the circle, R, does not change as the mass m moves in its orbit, the magnitude of the vector **v**, $|\mathbf{v}|$, must also be constant. The rate of change of the velocity vector is therefore determined only by the change in direction as m moves around the orbit, as shown in Figure 2.3. If we consider only small angles, $\delta\theta$, we can look at the way the vector **v** behaves by looking at Figure 2.2. The vector $\Delta\mathbf{v}$ is the change in direction of the velocity vector **v**. Note that this vector, $\Delta\mathbf{v}$, always points toward the mass M at the origin of the coordinate system. Thus,

$$\mathbf{F}_c = m\ \frac{d\mathbf{v}}{dt} = -m|\mathbf{v}|\frac{d\theta}{dt}\,\hat{r} \tag{2.3}$$

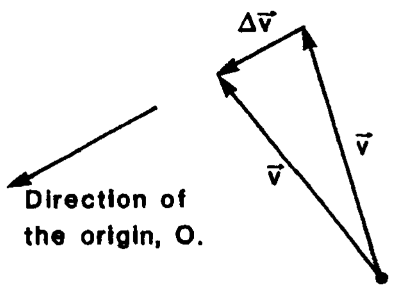

FIGURE 2.3

where the differential change in velocity is given by

$$d\mathbf{v} = -|\mathbf{v}|\, d\theta\, \hat{r} \tag{2.4}$$

The *angular velocity* ω of the mass *m* is defined as *dθ/dt* so that the centrifugal force can be rewritten as

$$\mathbf{F}_c = -m\, |\mathbf{v}|\, \omega \hat{r} \tag{2.5}$$

and simply writing v for $|\mathbf{v}|$ and recognizing that for a circular orbit of radius R the velocity v is

$$v = R\omega \tag{2.6}$$

we have, for the centrifugal force,

$$\mathbf{F}_c = -\frac{mv^2}{R}\hat{r} \tag{2.7}$$

Equating (2.7) to (2.1), we have

$$\mathbf{F}_G = \mathbf{F}_c \qquad G\frac{Mm}{R^2} = \frac{mv^2}{R} \tag{2.8}$$

Equation (2.8) allows determination of the orbital speed v (which is the magnitude) of the velocity vector **v** as

$$v = \sqrt{\frac{GM}{R}} \tag{2.9}$$

Note that the mass *m* appears on both sides of equation (2.8) so that the orbital speed is a function only of the radius of the circle and the magnitude of the mass *M*. Equation (2.9) can be rewritten in terms of the angular velocity defined in equation (2.6):

$$R\omega = \sqrt{\frac{GM}{R}} \tag{2.10}$$

and so we have

$$R^3\omega^2 = GM \tag{2.11}$$

The angular velocity can be related to the period of the orbit, that is, the time it takes to execute one orbit, by returning to the definition of the angular velocity,

$$\omega = \frac{d\theta}{dt} \quad \text{or} \quad \omega\, dt = d\theta \tag{2.12}$$

and integrating around one orbit,

$$\int_0^T \omega\, dt = \int_0^{2\pi} d\theta$$

we obtain

$$\omega T = 2\pi \tag{2.13}$$

where T is defined as the orbital period. Substituting equation (2.13) into (2.11) yields

$$R^3 = T^2 \left(\frac{GM}{4\pi^2} \right) \tag{2.14}$$

This statement is the third law of planetary motion as stated by Kepler (see Chapter 1) for the special case of circular orbits. It is obvious that the second law is also fulfilled for circular orbits since the orbital speed v is constant so that equal areas are swept out in equal time. In subsequent chapters, we shall show that these statements are valid for elliptic orbits as well.

Equation (2.11) is a very good approximation to the exact relation when we consider the motion of a satellite around Earth in a circular orbit. The approximate result assumes that the mass of the satellite can be neglected when compared to the mass of the central body. The derivation of the exact relation for circular motion utilizes Figure 2.4.

The satellite and Earth are moving around the center of mass of the Earth–satellite system. Since the mass of Earth is always many orders of magnitude larger than the mass of the satellite, the center of mass of the system is at the center of Earth for all practical purposes. As another interesting example, consider a binary star or a binary asteroid where two stars or two asteroids with comparable masses are revolving around each other. The distances from the center of mass are r_1 and r_2; the masses are

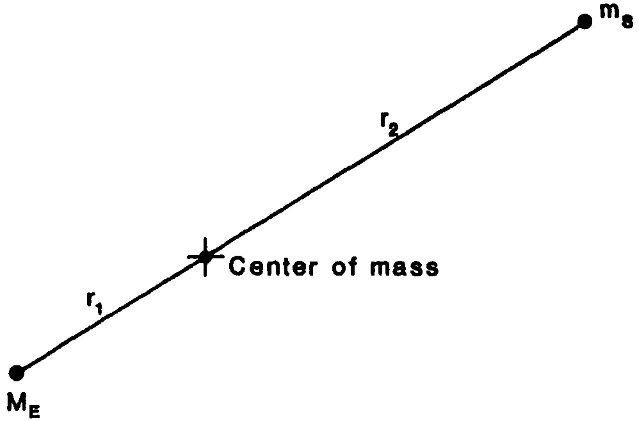

FIGURE 2.4 Motion of a satellite around the center of mass of the system.

m_1 and m_2. In our original problem, $m_1 = M_E$ and $m_2 = m_S$. The forces acting on the satellite are balanced if

$$m_s\omega^2 r_2 = G\frac{M_E m_S}{r^2} \quad (2.15)$$

where $r = r_1 + r_2$ since that is the total distance between the two interacting bodies.

The corresponding equation for Earth is

$$M_E\omega^2 r_1 = G\frac{M_E m_S}{r^2} \quad (2.16)$$

In the first equation m_S and in the second equation M_E are canceled. Since the center of mass is fixed in the system, we have

$$m_S r_2 = M_E r_1 \quad (2.17)$$

Adding the two previous equations (2.16) and (2.17), we have

$$(r_1 + r_2)\omega^2 = G\frac{M_E m_S}{r^2} \quad (2.18)$$

Equation (2.18) may be written as

$$\omega^2 r^3 = G(M_E + m_S) \quad (2.19)$$